

# CRITICAL SYSTEM INVOLVING FRACTIONAL LAPLACIAN

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ABSTRACT. In this paper, we discuss the existence of solutions for the following non-local critical systems:

$$\begin{cases} (-\Delta)^s u = \mu_1 |u|^{2^*-2} u + \frac{\alpha\gamma}{2^*} |u|^{\alpha-2} u |v|^\beta & \text{in } R^N, \\ (-\Delta)^s v = \mu_2 |v|^{2^*-2} v + \frac{\beta\gamma}{2^*} |u|^\alpha |v|^{\beta-2} v & \text{in } R^N, \\ u, v \in D_s(R^N). \end{cases}$$

By using the Nehari manifold, under proper conditions, we establish the existence and nonexistence of positive least energy solution.

## 1. INTRODUCTION

Equations and systems, involving laplacian have been studied for many years and are applied to many subjects, such as physical, chemical and materials science. Recently, a great attention has been focused on the study of equations and systems, involving fractional laplacian (see [3,4,5,8,10,11,12,15,16,18,19,22] and references therein). For the case of single equation, E. Colorado, A. de Pablo and U. Sánchez, in [3] studied the following nonhomogeneous single equation involving fractional laplacian,

$$\begin{cases} (-\Delta)^s u = \lambda u^q + u^{\frac{N+s}{N-s}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and proved the existence and multiplicity of solutions for suitable conditions depended on  $s$  and  $q$ . Furthermore in [4], they studied the following nonhomogeneous fractional equation involving critical Sobolev exponent,

$$\begin{cases} (-\Delta)^s u = |u|^{2^*-2} u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and proved the existence and multiplicity of solutions under appropriate conditions on the size of  $f$ . In [5], F.Tao, X.Wu studied the existence and multiplicity of positive solutions for fractional schödinger equations

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with critical growth. L.Caffarelli, L. Silvestre in [10] studied an extension problem related to the fractional laplacian, by given a new formulation of the fractional laplacian, and this idea was used in recent articles (see [12,17] and references therein). For more details about the results of single fractional equation, see [7,19]. Systems involving fractional laplacian in a bounded domain of  $R^N$  have also been studied in recently years. In [21], Z.Chen, W.Zou have studied the positive least energy solution, and phase separation for coupled systems schrödinger equation with critical exponent. Futhermore, Z.Chen, W.Zou in [20] gave the optimal constant for the existence of least energy solution of a coupled schrödinger system. By using of the idea of s-harmonic, Q.Wang in [12] studied the positive least energy solutions of fractional laplacian systems with critical exponent. X.He, M.Squassina and W.Zou in [17] have studied the Nehari manifold for fractional laplacian systems involving critical nonlinearities, and obtained the least energy solutions. For coupled systems involving fractional laplacian. Z.Guo and W.Zou in [22] showed that the positive solution has to be radially symmetric respect to some points in  $R^N$  and decays at infinity with certain rates. For the case of systems involving fractional laplacian in  $R^N$ . X.Zhang, J.Wang in [18] studied the symmetry results for systems involving fractional laplacian. In this paper, we study the following fractional laplacian system with critical exponent.

$$(1.1) \quad \begin{cases} (-\Delta)^s u = \mu_1 |u|^{2^*-2} u + \frac{\alpha\gamma}{2^*} |u|^{\alpha-2} u |v|^\beta & \text{in } R^N, \\ (-\Delta)^s v = \mu_2 |v|^{2^*-2} v + \frac{\beta\gamma}{2^*} |u|^\alpha |v|^{\beta-2} v & \text{in } R^N, \\ u, v \in D_s(R^N), \end{cases}$$

where, the fractional laplacian  $(-\Delta)^s$  is defined by

$$(-\Delta)^s u(x) = C(N, s) P.V. \int_{R^N} \frac{u(x) - u(y)}{|y - x|^{N+2s}} dy,$$

with

$$C(N, s) = \left( \int_{R^N} \frac{1 - \cos(\zeta)}{|\zeta|^{N+2s}} d\zeta \right)^{-1} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2-s)} s(1-s),$$

and

$$0 < s < 1, \mu_1, \mu_2 > 0, 2^* = \frac{2N}{N-2s}, N > 2s, \alpha, \beta > 1, \alpha + \beta = 2^*.$$

In this paper, by giving the range of  $\gamma$  and the relation between  $N$  and  $s$ , we get the existence and nonexistence of least energy solution of (1.1). The existence of positive least energy solution of (1.1) is

strongly depended on the following system.

$$(1.2) \quad \begin{cases} \mu_1 k^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} k^{\frac{\alpha-2}{2}} l^{\frac{\beta}{2}} = 1, \\ \mu_2 l^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} = 1, \\ k, l > 0. \end{cases}$$

Our main results are the following:

**Theorem 1.1.** *If  $\gamma < 0$ , then  $A = \frac{s}{N}(\mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}})S_s^{\frac{N}{2s}}$  and  $A$  is not attained.*

**Theorem 1.2.** *If  $2s \leq N \leq 4s$ ,  $\alpha, \beta > 2$  and*

$$(1.3) \quad 0 < \gamma \leq \frac{4Ns}{(N-2s)^2} \min\left\{\frac{\mu_1}{\alpha} \left(\frac{\alpha-2}{\beta-2}\right)^{\frac{\beta-2}{2}}, \frac{\mu_2}{\beta} \left(\frac{\alpha-2}{\beta-2}\right)^{\frac{\alpha-2}{2}}\right\},$$

*or  $N > 4s$ ,  $1 < \alpha, \beta < 2$  and*

$$(1.4) \quad \gamma \geq \frac{4Ns}{(N-2s)^2} \max\left\{\frac{\mu_1}{\alpha} \left(\frac{2-\beta}{2-\alpha}\right)^{\frac{2-\beta}{2}}, \frac{\mu_2}{\beta} \left(\frac{2-\alpha}{2-\beta}\right)^{\frac{2-\alpha}{2}}\right\},$$

*then  $A = \frac{s}{N}(k_0 + l_0)S_s^{\frac{N}{2s}}$ , and  $A$  is attained by  $(\sqrt{k_0}U_{\varepsilon,y}, \sqrt{l_0}U_{\varepsilon,y})$ , where  $(k_0, l_0)$  satisfies (1.2) and  $k_0 = \min\{k, (k, l) \text{ satisfies (1.2)}\}$ .*

**Theorem 1.3.** *Assume,  $N > 4s$  and  $1 < \alpha, \beta < 2$  hold, there exists a*

$$\gamma_1 \in (0, \frac{4Ns}{(N-2s)^2} \max\left\{\frac{\mu_1}{\alpha} \left(\frac{2-\beta}{2-\alpha}\right)^{\frac{2-\beta}{2}}, \frac{\mu_2}{\beta} \left(\frac{2-\alpha}{2-\beta}\right)^{\frac{2-\alpha}{2}}\right\},$$

*such that, for any  $\gamma \in (0, \gamma_1)$ , there exists a solution  $(k(\gamma), l(\gamma))$  of (1.2), satisfying  $E((\sqrt{k(\gamma)}U_{\varepsilon,y}, \sqrt{l(\gamma)}U_{\varepsilon,y})) > A$ , and  $(\sqrt{k(\gamma)}U_{\varepsilon,y}, \sqrt{l(\gamma)}U_{\varepsilon,y})$  is a positive solution of (1.1).*

**Remark 1.1.** Before presenting our next result, we would like to mention that the following fractional Brezis-Nirenberg problem

$$(-\Delta)^s u + \lambda u = |u|^{2^*-2}u, \quad x \in \Omega,$$

$$u = 0, \quad x \in \partial\Omega,$$

has a positive least energy solution  $U_\lambda$  for  $-\lambda_s(\Omega) < \lambda < 0$  (see[2]).

Consider the following system:

$$(1.5) \quad \begin{cases} (-\Delta)^s u + \lambda_1 u = \mu_1 |u|^{2^*-2}u + \frac{\alpha\gamma}{2^*} |u|^{\alpha-2}u |v|^\beta & \text{in } \Omega, \\ (-\Delta)^s v + \lambda_2 v = \mu_2 |v|^{2^*-2}v + \frac{\beta\gamma}{2^*} |u|^\alpha |v|^{\beta-2}v & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is an open bounded domain in  $R^N$ ,  $\lambda_1, \lambda_2 \in (-\lambda_s(\Omega), 0)$ , here  $\lambda_s(\Omega)$  is the first eigenvalue of  $(-\Delta)^s$  with Dirichlet boundary condition. By the idea of s-harmonic extension in [10] and the arguments in [12], we can show the following result:

If  $-\lambda_s(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0$ ,  $2s \leq N \leq 4s$ ,  $\alpha, \beta > 2$  and

$$0 < \gamma \leq \frac{4Ns}{(N-2s)^2} \min\left\{\frac{\mu_1}{\alpha} \left(\frac{\alpha-2}{\beta-2}\right)^{\frac{\beta-2}{2}}, \frac{\mu_2}{\beta} \left(\frac{\alpha-2}{\beta-2}\right)^{\frac{\alpha-2}{2}}\right\},$$

or  $-\lambda_s(\Omega) < \lambda_1 = \lambda_2 = \lambda < 0$ ,  $N > 4s$ ,  $1 < \alpha, \beta < 2$  and

$$\gamma \geq \frac{4Ns}{(N-2s)^2} \max\left\{\frac{\mu_1}{\alpha} \left(\frac{2-\beta}{2-\alpha}\right)^{\frac{2-\beta}{2}}, \frac{\mu_2}{\beta} \left(\frac{2-\alpha}{2-\beta}\right)^{\frac{2-\alpha}{2}}\right\},$$

then  $(\sqrt{k_0}U_\lambda, \sqrt{l_0}U_\lambda)$  is a positive least energy of (1.5), where  $(k_0, l_0)$  satisfies (1.2) and  $k_0 = \min\{k, (k, l) \text{ satisfies (1.2)}\}$ .

The paper is organized as follows. In section 2 we introduce some preliminaries and some Lemmas that will be used to prove Theorems 1.1, 1.2, 1.3. In section 3 we give two Propositions. In section 4 we prove Theorem 1.1. In section 5 we prove Theorem 1.2 and in section 6 we prove Theorem 1.3.

## 2. SOME PRELIMINARIES

Let  $D_s(R^N)$  be Hilbert space as the completion of  $C_c^\infty(R^N)$  equipped with the norm

$$\|u\|_{D_s(R^N)}^2 = \frac{C(N, s)}{2} \int_{R^N} \int_{R^N} \frac{|u(x) - u(y)|^2}{|y - x|^{N+2s}} dx dy,$$

and

$$(2.1) \quad S_s = \inf_{u \in D_s(R^N) \setminus 0} \frac{\|u\|_{D_s(R^N)}^2}{\left(\int_{R^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}},$$

be the sharp imbedding constant of  $D_s(R^N) \hookrightarrow L^{2^*}(R^N)$  and  $S_s$  is attained only on  $R^N$ . By [3],  $S_s$  is attained by  $\tilde{u} = \kappa(\varepsilon^2 + |x - x_0|)^{-\frac{N-2s}{2}}$ , that is

$$S_s = \frac{\|\tilde{u}\|_{D_s(R^N)}^2}{\left(\int_{R^N} |\tilde{u}|^{2^*} dx\right)^{\frac{2}{2^*}}}.$$

Let

$$\bar{u} = \frac{\tilde{u}}{|\tilde{u}|_{2^*}},$$

by Lemma 2.12 in [22],  $u_\mu = (\frac{S_s}{\mu})^{\frac{1}{2^*-2}} \bar{u}$  is a positive ground state solution of

$$(2.2) \quad (-\Delta)^s u = \mu |u|^{2^*-2} u, \text{ in } R^N$$

and  $U_{\varepsilon,y} = (S_s)^{\frac{1}{2^*-2}} \bar{u}$  is a positive ground state solution of

$$(2.3) \quad (-\Delta)^s u = |u|^{2^*-2} u, \text{ in } R^N,$$

with

$$(2.4) \quad \|U_{\varepsilon,y}\|_{D_s(R^N)}^2 = \int_{R^N} |U_{\varepsilon,y}|^{2^*} dx = (S_s)^{\frac{N}{2s}}.$$

The energy function associated with (1.1) is given by

$$E(u, v) = \frac{1}{2} (\|u\|_{D_s(R^N)}^2 + \|v\|_{D_s(R^N)}^2) - \frac{1}{2^*} \int_{R^N} \mu_1 |u|^{2^*} + \mu_2 |v|^{2^*} + \gamma |u|^\alpha |v|^\beta dx.$$

Define the Nehari manifold

$$\begin{aligned} \mathbb{N} &= \{(u, v) \in D_s(R^N) \times D_s(R^N) : u \neq 0, v \neq 0, \\ &\quad \|u\|_{D_s(R^N)}^2 = \int_{R^N} \mu_1 |u|^{2^*} + \frac{\alpha\gamma}{2^*} |u|^\alpha |v|^\beta dx, \\ &\quad \|v\|_{D_s(R^N)}^2 = \int_{R^N} \mu_2 |v|^{2^*} + \frac{\beta\gamma}{2^*} |u|^\alpha |v|^\beta dx\}. \end{aligned}$$

$$\begin{aligned} A &= \inf_{(u,v) \in \mathbb{N}} E(u, v) = \inf_{(u,v) \in \mathbb{N}} \frac{s}{N} (\|u\|_{D_s(R^N)}^2 + \|v\|_{D_s(R^N)}^2) \\ &= \inf_{(u,v) \in \mathbb{N}} \frac{s}{N} \int_{R^N} \mu_1 |u|^{2^*} + \mu_2 |v|^{2^*} + \gamma |u|^\alpha |v|^\beta dx. \end{aligned}$$

If  $(u, v) \neq (0, 0)$  and  $u \neq 0, v \neq 0$ , then we claim  $(u, v)$  is a nontrivial solution of (1.1).

Obviously,  $\mathbb{N} \neq \emptyset$ , any nontrivial solution of (1.1) is in  $\mathbb{N}$ .

**Lemma 2.1.** *If  $(u, v) \in \mathbb{N}$  is any nontrivial solution of (1.1), then*

$$\|u\|_{D_s(R^N)}^2 > 0, \quad \|v\|_{D_s(R^N)}^2 > 0.$$

*Proof.* If

$$\|u\|_{D_s(R^N)}^2 = 0, \quad \|v\|_{D_s(R^N)}^2 = 0,$$

then by the definition of  $S_s$ , we have

$$\int_{R^N} |u|^{2^*} dx \leq S_s^{-\frac{2^*}{2}} (\|u\|_{D_s(R^N)}^2)^{\frac{2^*}{2}} = 0.$$

Thus

$$u \equiv 0, v \equiv 0, \text{ a.e. } x \in R^N,$$

which contradicts to  $(u, v) \neq (0, 0)$ .  $\square$

The following Lemma 2.2 show if the energy functional attained its minimum at some points  $(u, v) \in \mathbb{N}$ , then  $(u, v)$  is a solution of (1.1). The main idea is use lagrange multiplier method to prove Lemma 2.2.

**Lemma 2.2.** *Assume  $\gamma < 0$ , if  $A$  is attained by a couple  $(u, v) \in \mathbb{N}$ , then  $(u, v)$  is a solution of (1.1).*

*Proof.* Define

$$\begin{aligned} \mathbb{N}_1 &= \{(u, v) \in D_s(R^N) \times D_s(R^N) : u \neq 0, v \neq 0 \\ G_1(u, v) &= \|u\|_{D_s(R^N)}^2 - \int_{R^N} \mu_1 |u|^{2^*} + \frac{\alpha\gamma}{2^*} |u|^\alpha |v|^\beta dx = 0\}, \\ \mathbb{N}_2 &= \{(u, v) \in D_s(R^N) \times D_s(R^N) : u \neq 0, v \neq 0 \\ G_2(u, v) &= \|u\|_{D_s(R^N)}^2 - \int_{R^N} \mu_1 |u|^{2^*} + \frac{\alpha\gamma}{2^*} |u|^\alpha |v|^\beta dx = 0\}. \end{aligned}$$

Then  $\mathbb{N} = \mathbb{N}_1 \cap \mathbb{N}_2$ . By the definition of Fréchet derivative, we have

$$\begin{aligned} E'(u, v)(\varphi_1, \varphi_2) &= (u, \varphi_1) + (v, \varphi_2) \\ &\quad - \int_{R^N} \mu_1 |u|^{2^*-2} u \varphi_1 + \mu_2 |v|^{2^*-2} v \varphi_2 \\ &\quad - \frac{\alpha\gamma}{2^*} \int_{R^N} \alpha |u|^{\alpha-2} u \varphi_1 |v|^\beta + \beta |u|^\alpha |v|^{\beta-2} v \varphi_2 dx, \\ G'_1(u, v)(\varphi_1, \varphi_2) &= 2(u, \varphi_1) - 2^* \int_{R^N} \mu_1 |u|^{2^*-2} u \varphi_1 dx \\ &\quad - \frac{\alpha\gamma}{2^*} \int_{R^N} \alpha |u|^{\alpha-2} u \varphi_1 |v|^\beta + \beta |u|^\alpha |v|^{\beta-2} v \varphi_2 dx, \\ G'_2(u, v)(\varphi_1, \varphi_2) &= 2(v, \varphi_2) - 2^* \int_{R^N} \mu_2 |v|^{2^*-2} v \varphi_2 dx \\ &\quad - \frac{\beta\gamma}{2^*} \int_{R^N} \alpha |u|^{\alpha-2} u \varphi_1 |v|^\beta + \beta |u|^\alpha |v|^{\beta-2} v \varphi_2 dx, \end{aligned}$$

where

$$(u, \varphi_1) = \frac{C(N, \alpha)}{2} \int_{R^N} \int_{R^N} \frac{(u(x) - u(y))(\varphi_1(x) - \varphi_1(y))}{|y - x|^{N+2s}} dx dy.$$

Then

$$E'(u, v)(u, 0) = G_1(u, v) = 0,$$

$$E'(u, v)(0, v) = G_2(u, v) = 0,$$

$$G'_1(u, v)(u, 0) = 2\|u\|_{D_s(R^N)}^2 - 2^* \int_{R^N} \mu_1 |u|^{2^*} dx - \frac{\alpha\gamma}{2^*} \int_{R^N} \alpha |u|^\alpha |v|^\beta dx$$

$$= -(2^* - 2) \int_{R^N} \mu_1 |u|^{2^*} dx + (2 - \alpha) \int_{R^N} \frac{\alpha\gamma}{2^*} |u|^\alpha |v|^\beta dx,$$

$$G'_1(u, v)(0, v) = 2\|v\|_{D_s(R^N)}^2 - 2^* \int_{R^N} \mu_2 |v|^{2^*} dx - \frac{\beta\gamma}{2^*} \int_{R^N} \beta |u|^\alpha |v|^\beta dx$$

$$= -(2^* - 2) \int_{R^N} \mu_2 |v|^{2^*} dx + (2 - \beta) \int_{R^N} \frac{\beta\gamma}{2^*} |u|^\alpha |v|^\beta dx,$$

$$G'_1(u, v)(0, v) = -\frac{\alpha\gamma}{2^*} \int_{R^N} \beta |u|^\alpha |v|^\beta dx > 0,$$

$$G'_2(u, v)(u, 0) = -\frac{\beta\gamma}{2^*} \int_{R^N} \alpha |u|^\alpha |v|^\beta dx > 0.$$

Suppose that  $(u, v) \in \mathbb{N}$  is a minimizer for  $E$  restricted to  $\mathbb{N}$ , by the standard minimization theory, there exist two lagrange multipliers  $L_1, L_2 \in \mathbb{R}$  such that,

$$E'(u, v) + L_1 G'_1(u, v) + L_2 G'_2(u, v) = 0.$$

Thus we have

$$L_1 G'_1(u, 0) + L_2 G'_2(u, 0) = 0,$$

$$L_1 G'_1(0, v) + L_2 G'_2(0, v) = 0,$$

and

$$G'_1(u, v)(u, 0) + G'_1(u, v)(0, v) = -(2^* - 2)\|u\|_{D_s(R^N)}^2 \leq 0,$$

$$G'_2(u, v)(u, 0) + G'_2(u, v)(0, v) = -(2^* - 2)\|v\|_{D_s(R^N)}^2 \leq 0.$$

By Lemma 2.1, we have

$$\|u\|_{D_s(R^N)}^2 > 0, \quad \|v\|_{D_s(R^N)}^2 > 0,$$

hence

$$|G'_1(u, v)(u, 0)| = -G'_1(u, v)(u, 0) > G'_1(u, v)(0, v),$$

$$|G'_2(u, v)(0, v)| = -G'_2(u, v)(0, v) > G'_2(u, v)(u, 0).$$

Define the matrix

$$M = \begin{pmatrix} G'_1(u, v)(u, 0) & G'_2(u, v)(u, 0) \\ G'_1(u, v)(0, v) & G'_2(u, v)(0, v) \end{pmatrix},$$

then

$$\det(M) = |G'_1(u, v)(u, 0)| |G'_2(u, v)(0, v)| - G'_1(u, v)(0, v) G'_2(u, v)(u, 0) > 0,$$

which means  $L_1 = L_2 = 0$ , that is  $E'(u, v) = 0$ . □

Define functions

$$(2.5) \quad \begin{cases} F_1(k, l) = \mu_1 k^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} k^{\frac{\alpha-2}{2}} l^{\frac{\beta}{2}} - 1 & k > 0, l \geq 0, \\ F_2(k, l) = \mu_2 l^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} - 1 & k \geq 0, l > 0, \\ l(k) = \left(\frac{2^*}{\alpha\gamma}\right)^{\frac{2}{\beta}} k^{\frac{2-\alpha}{\beta}} (1 - \mu_1 k^{\frac{2^*-2}{2}})^{\frac{2}{\beta}} & 0 < k \leq \mu_1^{-\frac{2}{2^*-2}}, \\ k(l) = \left(\frac{2^*}{\beta\gamma}\right)^{\frac{2}{\alpha}} l^{\frac{2-\beta}{\alpha}} (1 - \mu_2 l^{\frac{2^*-2}{2}})^{\frac{2}{\alpha}} & 0 < l \leq \mu_2^{-\frac{2}{2^*-2}}, \end{cases}$$

then

$$F_1(k, l(k)) \equiv 0, \quad F_2(k(l), l) \equiv 0.$$

**Remark 2.1.** If  $(k, l)$  satisfies (1.2), then  $(\sqrt{k}U_{\varepsilon, y}, \sqrt{l}U_{\varepsilon, y})$  is a solution of (1.1), where  $U_{\varepsilon, y}$  satisfy (2.3) and (2.4). Hence the main work is to establish the conditions for the existence of solutions of (1.2).

In order to prove the existence of solution of (1.2), we present the following Lemma 2.3.

**Lemma 2.3.** Assume that  $N > 4s, 1 < \alpha, \beta < 2, \gamma > 0$ , then

$$(2.6) \quad F_1(k, l) = 0, \quad F_2(k, l) = 0 \quad k, l > 0,$$

has a solution  $(k_0, l_0)$  such that

$$(2.7) \quad F_2(k, l(k)) < 0 \quad \forall k \in (0, k_0),$$

where  $(k_0, l_0)$  satisfies  $k_0 = \min\{k : (k, l) \text{ satisfies (1.2)}\}$ .

Similarly, (2.6) has a solution  $(k_1, l_1)$ , such that

$$(2.8) \quad F_1(k(l), l) < 0 \quad \forall l \in (0, l_1).$$

*Proof.* By  $F_1(k, l) = 0 \quad k, l > 0$ , then

$$l = l(k) \quad \forall k \in (0, \mu_1^{-\frac{2}{2^*-2}}).$$

Substituting this into  $F_2(k, l) = 0$ , we have

$$(2.9) \quad \begin{aligned} & \mu_2 \left(\frac{2^*}{\alpha\gamma}\right)^{\frac{\alpha}{\beta}} (1 - \mu_1 k^{\frac{2^*-2}{2}})^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{2^*} k^{-\frac{(2^*-2)\alpha}{2\beta}} \\ & - \left(\frac{2^*}{\alpha\gamma}\right)^{\frac{2-\alpha}{\beta}} k^{-\frac{(2^*-2)(2-\beta)}{2\beta}} (1 - \mu_1 k^{\frac{2^*-2}{2}})^{\frac{2-\beta}{\beta}} = 0. \end{aligned}$$



Let

$$f(k) = \mu_2 \left( \frac{2^*}{\alpha\gamma} \right)^{\frac{\alpha}{\beta}} (1 - \mu_1 k^{\frac{2^*-2}{2}})^{\frac{\alpha}{\beta}} + \frac{\beta\gamma}{2^*} k^{-\frac{(2^*-2)\alpha}{2\beta}} - \left( \frac{2^*}{\alpha\gamma} \right)^{\frac{2-\alpha}{\beta}} k^{-\frac{(2^*-2)(2-\beta)}{2\beta}} (1 - \mu_1 k^{\frac{2^*-2}{2}})^{\frac{2-\beta}{\beta}},$$

then (2.9) has a solution is equivalent to  $f(k) = 0$  has a solution in  $(0, \mu_1^{-\frac{2}{2^*-2}})$ . Since  $1 < \alpha, \beta < 2$ , we get that

$$\lim_{k \rightarrow 0^+} f(k) = -\infty, \quad f(\mu_1^{-\frac{2}{2^*-2}}) = \frac{\beta\gamma}{2^*} \mu_1^{-\frac{\alpha}{\beta}} > 0,$$

then using the intermediate value theorem, we know that, there exists

$$k_0 \in (0, \mu_1^{-\frac{2}{2^*-2}}) \quad \text{s.t.} \quad f(k_0) = 0,$$

and

$$f(k) < 0 \quad \forall k \in (0, k_0).$$

Let  $l_0 = l(k_0)$ , then  $(k_0, l_0)$  is the solution of (2.6) and satisfies (2.7).  $\square$

**Remark 2.2.** It is easy to see, if  $(\sqrt{k_0}U_{\varepsilon,y}, \sqrt{l_0}U_{\varepsilon,y}) \in \mathbb{N}$ , then

$$A \leq E(\sqrt{k_0}U_{\varepsilon,y}, \sqrt{l_0}U_{\varepsilon,y}) = \frac{s}{N}(k_0 + l_0)S_s^{\frac{N}{2s}}.$$

In order to prove  $A \geq \frac{s}{N}(k_0 + l_0)S_s^{\frac{N}{2s}}$ , we give the following Lemma 2.4 and Lemma 2.5

**Lemma 2.4.** Assume  $N > 4s, 1 < \alpha, \beta < 2$  and (1.3) holds, then

$$l(k) + k \text{ is strictly increasing on } [0, \mu_1^{-\frac{2}{2^*-2}}],$$

$$k(l) + l \text{ is strictly increasing on } [0, \mu_2^{-\frac{2}{2^*-2}}].$$

*Proof.* Since

$$\begin{aligned} l'(k) &= \left( \frac{2^*}{\alpha\gamma} \right)^{\frac{2}{\beta}} \frac{2}{\beta} (k^{\frac{2-\alpha}{2}} - \mu_1 k^{\frac{\beta}{2}})^{\frac{2-\beta}{\beta}} \left( \frac{2-\alpha}{2} k^{-\frac{\alpha}{2}} - \frac{\mu_1\beta}{2} k^{\frac{\beta-2}{2}} \right) \\ &= \left( \frac{2^*\mu_1}{\alpha\gamma} \right)^{\frac{2}{\alpha}} k^{\frac{2-2^*}{\beta}} (\mu_1^{-1} - k^{\frac{2^*-2}{2}})^{\frac{2-\beta}{\beta}} \left( \frac{2-\alpha}{\mu_1\beta} - k^{\frac{2^*-2}{2}} \right), \\ k'(l) &= \left( \frac{2^*}{\beta\gamma} \right)^{\frac{2}{\alpha}} \frac{2}{\alpha} (l^{\frac{2-\beta}{2}} - \mu_2 l^{\frac{\alpha}{2}})^{\frac{2-\alpha}{\alpha}} \left( \frac{2-\beta}{2} l^{-\frac{\beta}{2}} - \frac{\mu_2\alpha}{2} l^{\frac{\alpha-2}{2}} \right) \\ &= \left( \frac{2^*\mu_2}{\beta\gamma} \right)^{\frac{2}{\alpha}} l^{\frac{2-2^*}{\alpha}} (\mu_2^{-1} - l^{\frac{2^*-2}{2}})^{\frac{2-\alpha}{\alpha}} \left( \frac{2-\beta}{\mu_2\alpha} - l^{\frac{2^*-2}{2}} \right), \end{aligned}$$

hence

$$l'((\frac{2-\alpha}{\mu_1\beta})^{\frac{2}{2^*-2}}) = l'(\mu_1^{-\frac{2}{2^*-2}}) = 0,$$

$$k'((\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}}) = k'(\mu_2^{-\frac{2}{2^*-2}}) = 0$$

and

$$l'(k) > 0 \Leftrightarrow k \in (0, (\frac{2-\alpha}{\mu_1\beta})^{\frac{2}{2^*-2}}),$$

$$l'(k) < 0 \Leftrightarrow k \in ((\frac{2-\alpha}{\mu_1\beta})^{\frac{2}{2^*-2}}, \mu_1^{-\frac{2}{2^*-2}}),$$

$$k'(l) > 0 \Leftrightarrow l \in (0, (\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}}),$$

$$k'(l) < 0 \Leftrightarrow l \in ((\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}}, \mu_2^{-\frac{2}{2^*-2}}).$$

Since

$$l''(k) = \frac{2-\beta}{\beta} (\frac{2^*\mu_1}{\alpha\gamma})^{\frac{2}{\beta}} k^{\frac{2-2\beta-\alpha}{\beta}} (\mu_1^{-1} - k^{\frac{2^*-2}{2}})^{\frac{2-2\beta}{\beta}}$$

$$\times [(\frac{2-\alpha}{\mu_1\beta} - k^{\frac{2^*-2}{2}})^2 - (\mu_1^{-1} - k^{\frac{2^*-2}{2}})(\frac{\alpha(2-\alpha)}{\mu_1\beta(2-\beta)} - k^{\frac{2^*-2}{2}})],$$

$$k''(l) = \frac{2-\alpha}{\alpha} (\frac{2^*\mu_2}{\beta\gamma})^{\frac{2}{\alpha}} l^{\frac{2-2\alpha-\beta}{\alpha}} (\mu_2^{-1} - l^{\frac{2^*-2}{2}})^{\frac{2-2\alpha}{\alpha}}$$

$$\times [(\frac{2-\beta}{\mu_2\alpha} - l^{\frac{2^*-2}{2}})^2 - (\mu_2^{-1} - l^{\frac{2^*-2}{2}})(\frac{\beta(2-\beta)}{\mu_2\alpha(2-\alpha)} - l^{\frac{2^*-2}{2}})],$$

we have

$$l''(k) = 0 \Leftrightarrow k = (\frac{2(2-\alpha)}{\mu_1\beta(4-2^*)})^{\frac{2}{2^*-2}} \triangleq k_1 \in ((\frac{2-\alpha}{\mu_1\beta})^{\frac{2}{2^*-2}}, \mu_1^{-\frac{2}{2^*-2}}),$$

$$k''(l) = 0 \Leftrightarrow l = (\frac{2(2-\beta)}{\mu_2\alpha(4-2^*)})^{\frac{2}{2^*-2}} \triangleq l_1 \in ((\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}}, \mu_2^{-\frac{2}{2^*-2}}).$$

Then by (1.4) we obtain

$$l'(k)_{min} = l'(k_1) = -(\frac{2^*(2^*-2)\mu_1}{2\alpha\gamma})^{\frac{2}{\beta}} (\frac{2-\beta}{2-\alpha})^{\frac{2-\beta}{\beta}} \geq -1,$$

$$k'(l)_{min} = k'(l_1) = -(\frac{2^*(2^*-2)\mu_2}{2\beta\gamma})^{\frac{2}{\alpha}} (\frac{2-\alpha}{2-\beta})^{\frac{2-\alpha}{\alpha}} \geq -1.$$

Hence

$$l'(k) > -1, \forall k \in (0, \mu_1^{-\frac{2}{2^*-2}}),$$

$$k'(l) > -1, \forall l \in (0, \mu_2^{-\frac{2}{2^*-2}}),$$

which means  $l(k) + k$  is strictly increasing on  $[0, \mu_1^{-\frac{2}{2^*-2}}]$ , and  $k(l) + l$  is strictly increasing on  $[0, \mu_1^{-\frac{2}{2^*-2}}]$ .  $\square$

**Lemma 2.5.** Assume  $N > 4s$ ,  $1 < \alpha, \beta < 2$  and (1.3) hold, let  $(k_0, l_0)$  be the same as in Lemma 2.3. Then

$$(2.10) \quad (k_0 + l_0)^{\frac{2^*-2}{2}} \max\{\mu_1, \mu_2\} < 1$$

and

$$(2.11) \quad F_2(k, l(k)) < 0, \forall k \in (0, k_0); F_1(k(l), l) < 0, \forall l \in (0, l_0).$$

*Proof.* Since

$$k_0 < \mu_1^{-\frac{2}{2^*-2}},$$

by Lemma 2.4, we have,

$$\mu_1^{-\frac{2}{2^*-2}} = l(\mu_1^{-\frac{2}{2^*-2}}) + \mu_1^{-\frac{2}{2^*-2}} \geq l(k_0) + k_0 = l_0 + k_0.$$

That is

$$\mu_1(k_0 + l_0)^{\frac{2^*-2}{2}} \leq 1.$$

Similarly

$$\mu_2(k_0 + l_0)^{\frac{2^*-2}{2}} \leq 1.$$

Hence

$$(k_0 + l_0)^{\frac{2^*-2}{2}} \max\{\mu_1, \mu_2\} < 1.$$

To prove (2.11) by Lemma 2.3, we only needs to show that  $(k_0, l_0) = (k_1, l_1)$ . By (2.7), (2.8), we have  $l_0 \geq l_1$ ,  $k_1 \geq k_0$ . Suppose by contradiction that  $k_1 > k_0$ , then

$$l(k_1) + k_1 > l(k_0) + k_0,$$

hence

$$l_1 + k(l_1) = l(k_1) + k_1 > l(k_0) + k_0 = l_0 + k(l_0).$$

Since  $k(l) + l$  is strictly increasing for  $l \in [0, \mu_2^{-\frac{2}{2^*-2}}]$ , therefore

$$l_1 > l_0,$$

which contradicts to  $l_0 \geq l$ , then we have

$$k_1 = k_0.$$

Similarly

$$l_1 = l_0.$$

$\square$

Define

$$\begin{aligned} A_\epsilon &= \inf_{(u,v) \in \mathbb{N}_\epsilon} E_\epsilon(u, v) \\ E_\epsilon(u, v) &= \frac{1}{2} ||(u, v)||_{D_s}^2 \\ &\quad - \int_{B_1} (\mu_1 |u|^{2^*-2\epsilon} + \mu_2 |v|^{2^*-2\epsilon} + \gamma |u|^{\alpha-\epsilon} |v|^{\beta-\epsilon}) \end{aligned}$$

$$\begin{aligned} \mathbb{N}_\epsilon &= \{(u, v) \in D_s(B_1) \setminus \{(0, 0)\} : G_\epsilon(u, v) = ||(u, v)||_{D_s}^2 \\ &\quad - \int_{B_1} (\mu_1 |u|^{2^*-2\epsilon} + \mu_2 |v|^{2^*-2\epsilon} + \gamma |u|^{\alpha-\epsilon} |v|^{\beta-\epsilon}) = 0\} \end{aligned}$$

The following Lemma 2.6 was given in [22].

**Lemma 2.6.** [22] *If  $u_\mu = (\frac{S_s}{\mu})^{\frac{1}{2^*-2}} \bar{u}$  is a positive ground state solution of (2.2), then*

$$\begin{aligned} A &< \min\left\{ \inf_{(u,v) \in \mathbb{N}} E(u, 0), \inf_{(u,v) \in \mathbb{N}} E(0, v) \right\}, \\ &= \min\{E(u_{\mu_1}, 0), E(0, u_{\mu_2})\}, \\ &= \min\left\{ \frac{S}{N} \mu_1^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}, \frac{S}{N} \mu_2^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}} \right\} \end{aligned}$$

and

$$A_\epsilon = A = E(u, v),$$

### 3. TWO PROPOSITIONS

**Proposition 3.1.** *Assume  $c, d \in \mathbb{R}$  satisfy*

$$(3.1) \quad \begin{cases} \mu_1 k^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} k^{\frac{\alpha-2}{2}} l^{\frac{\beta}{2}} \geq 1, \\ \mu_2 l^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} \geq 1, \\ k, l > 0. \end{cases}$$

*If  $2s < N < 4s$ ,  $\alpha, \beta > 2$  and (1.3) hold, then  $c + d \geq k + l$ , where  $k, l \in \mathbb{R}$  satisfy (1.2).*

*Proof.* Let  $y = c + d, x = \frac{c}{d}, y_0 = k + l, x_0 = \frac{k}{l}$  we have

$$\begin{aligned} y^{\frac{2^*-2}{2}} &\geq \frac{(x+1)^{\frac{2^*-2}{2}}}{\mu_1 x^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} x^{\frac{\alpha-2}{2}}} \triangleq f_1(x), y_0^{\frac{2^*-2}{2}} = f_1(x_0), \\ y^{\frac{2^*-2}{2}} &\geq \frac{(x+1)^{\frac{2^*-2}{2}}}{\mu_2 + \frac{\beta\gamma}{2^*} x^{\frac{\alpha}{2}}} \triangleq f_2(x), y_0^{\frac{2^*-2}{2}} = f_2(x_0). \end{aligned}$$

Then,

$$\begin{aligned} f_1'(x) &= \frac{\alpha\gamma(x+1)^{\frac{2^*-4}{2}} x^{\frac{\alpha-4}{2}}}{22^*(\mu_1 x^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} x^{\frac{\alpha-2}{2}})^2} \left[ -\frac{2^*(2^*-2)\mu_1}{\alpha\gamma} x^{\frac{\beta}{2}} + \beta x - (\alpha-2) \right], \\ f_2'(x) &= \frac{\beta\gamma(x+1)^{\frac{2^*-4}{2}}}{22^*(\mu_2 + \frac{\beta\gamma}{2^*} x^{\frac{\alpha}{2}})^2} \left[ (\beta-2)x^{\frac{\alpha}{2}} - \alpha x^{\frac{\alpha-2}{2}} + \frac{2^*(2^*-2)\mu_2}{\beta\gamma} \right]. \end{aligned}$$

Let

$$\begin{aligned} g_1(x) &= -\frac{2^*(2^*-2)\mu_1}{\alpha\gamma} x^{\frac{\beta}{2}} + \beta x - (\alpha-2), \\ g_2(x) &= (\beta-2)x^{\frac{\alpha}{2}} - \alpha x^{\frac{\alpha-2}{2}} + \frac{2^*(2^*-2)\mu_2}{\beta\gamma}, \end{aligned}$$

then,

$$\begin{aligned} g_1'(x) &= 0 \Leftrightarrow -\frac{2^*(2^*-2)\mu_1}{\alpha\gamma} \frac{\beta}{2} x^{\frac{\beta-2}{2}} + \beta = 0, \\ &\Leftrightarrow x = \left( \frac{2\alpha\gamma}{2^*(2^*-2)\mu_1} \right)^{\frac{2}{\beta-2}} \triangleq x_1, \\ g_1'(x) &> 0 \Leftrightarrow x < \left( \frac{2\alpha\gamma}{2^*(2^*-2)\mu_1} \right)^{\frac{2}{\beta-2}}, \\ g_1'(x) &< 0 \Leftrightarrow x > \left( \frac{2\alpha\gamma}{2^*(2^*-2)\mu_1} \right)^{\frac{2}{\beta-2}}, \\ g_2'(x) &= 0 \Leftrightarrow x = \frac{\alpha-2}{\beta-2} \triangleq x_2, \\ g_2'(x) &> 0 \Leftrightarrow x > \frac{\alpha-2}{\beta-2}, \\ g_2'(x) &< 0 \Leftrightarrow x < \frac{\alpha-2}{\beta-2}. \end{aligned}$$

Therefore,  $g_1(x)$  is increasing in  $(0, (\frac{2\alpha\gamma}{2^*(2^*-2)\mu_1})^{\frac{2}{\beta-2}})$  and is decreasing in  $((\frac{2\alpha\gamma}{2^*(2^*-2)\mu_1})^{\frac{2}{\beta-2}}, +\infty)$ ,  $g_2(x)$  is decreasing in  $(0, \frac{\alpha-2}{\beta-2})$  and is increasing in  $(\frac{\alpha-2}{\beta-2}, +\infty)$ .

Hence

$$\begin{aligned} g_1(x)_{\max} &= g_1(x_1) = (\beta - 2) \left( \frac{2\alpha\gamma}{2^*(2^*-2)\mu_1} \right)^{\frac{2}{\beta-2}} - (\alpha - 2), \\ g_2(x)_{\min} &= g_2(x_2) = -2 \left( \frac{\alpha-2}{\beta-2} \right)^{\frac{\alpha-2}{2}} + \frac{2^*(2^*-2)\mu_2}{\beta\gamma}. \end{aligned}$$

Then by (1.3), we have

$$g_1(x)_{\max} \leq 0, \quad g_2(x)_{\min} \geq 0.$$

Therefore,  $f_1(x)$  is strictly decreasing in  $(0, +\infty)$ , and  $f_2(x)$  is strictly increasing in  $(0, +\infty)$ . Hence we have

$$\begin{aligned} y^{\frac{2^*-2}{2}} &\geq \max\{f_1(x), f_1(x)\}, \\ &\geq \min(\max\{f_1(x), f_1(x)\}), \\ &= \min_{f_1(x)=f_2(x)} (\max\{f_1(x), f_1(x)\}) = y_0^{\frac{2^*-2}{2}}, \end{aligned}$$

which means  $c + d \geq k + l$ . □

**Proposition 3.2.** Assume  $N > 4s, 1 < \alpha, \beta < 2$  and (1.3) holds, then

$$(3.2) \quad \begin{cases} k + l \leq k_0 + l_0, \\ F_1(k, l) \geq 0, \quad F_2(k, l) \geq 0, \\ k, l > 0, \quad (k, l) \neq (0, 0), \end{cases}$$

has an unique solution  $(k, l) = (k_0, l_0)$ .

*Proof.* Obviously  $(k_0, l_0)$  satisfies (3.2). Suppose  $\tilde{k}, \tilde{l}$  is another solution of (3.2). Without loss of generality, assume that  $\tilde{k} > 0$ , then  $\tilde{l} > 0$ . In fact, if  $\tilde{l} = 0$ , then

$$\tilde{k} \leq k_0 + l_0,$$

and

$$F_1(\tilde{k}, 0) = \mu_1 \tilde{k}^{\frac{2^*-2}{2}} - 1 \geq 0,$$

therefore

$$1 \leq \mu_1 \tilde{k}^{\frac{2^*-2}{2}} \leq \mu_1 (k_0 + l_0)^{\frac{2^*-2}{2}},$$

which contradicts with Lemma 2.5. In the following, we prove that  $\tilde{k} = k_0$ . Suppose by contradiction that  $\tilde{k} < k_0$ , by the proof of Lemma 2.4, we have  $k(l)$  is strictly increasing on  $(0, (\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}})$ , and strictly decreasing on  $((\frac{2-\beta}{\mu_2\alpha})^{\frac{2}{2^*-2}}, \mu_2^{-\frac{2}{2^*-2}})$ .

On the one hand, since

$$k(0) = k(\mu_2^{-\frac{2}{2^*-2}}) = 0,$$

and

$$0 < \tilde{k} < k_0,$$

therefore, there exist  $l_1, l_2$  satisfying

$$0 < l_1 < l_2 < \mu_2^{-\frac{2}{2^*-2}},$$

such that

$$k(l_1) = k(l_2) = \tilde{k}$$

and

$$(3.3) \quad F_2(\tilde{k}, l) < 0 \Leftrightarrow \tilde{k} < k(l) \Leftrightarrow l_1 < l < l_2.$$

Futhermore

$$F_1(\tilde{k}, \tilde{l}) \geq 0, \quad F_2(\tilde{k}, \tilde{l}) \geq 0,$$

we have

$$\tilde{l} > l(\tilde{k}), \quad \tilde{l} \leq l_1 \text{ or } \tilde{l} \geq l_2.$$

By (2.11), we see  $F_2(\tilde{k}, l(\tilde{k})) < 0$ , by (3.3) we get that,  $l_1 < l < l_2$ , therefore

$$\tilde{l} \geq l_2.$$

On the other hand, let  $l_3 = k_0 + l_0 - \tilde{k}$ , then  $l_3 > l_0$  and

$$k(l_3) + k_0 + l_0 - \tilde{k} = k(l_3) + l_3 > k(l_0) + l_0 = k_0 + l_0,$$

that is  $k(l_3) > \tilde{k}$ . By (3.3) we have  $l_1 < l_3 < l_2$ .

Since

$$\tilde{k} + \tilde{l} \leq k_0 + l_0,$$

we obtained that

$$\tilde{l} \leq k_0 + l_0 - \tilde{k} = l_3 < l_2.$$

This contradicts to  $\tilde{l} \geq l_2$ , the proof completes. □

## 4. PROOF OF THEOREM 1.1

*Proof.*

$$\begin{aligned} \text{For any } (u, v) \in \mathbb{N}, \|u\|_{D_s(R^N)}^2 &= \int_{R^N} \mu_1 |u|^{2^*} + \frac{\alpha\gamma}{2^*} |u|^\alpha |v|^\beta dx \\ &\leq \int_{R^N} \mu_1 |u|^{2^*} dx \leq \mu_1 S_s^{-\frac{2^*}{2}} (\|u\|_{D_s(R^N)}^2)^{\frac{2^*}{2}}. \end{aligned}$$

Therefore

$$\|u\|_{D_s(R^N)}^2 \geq \mu_1^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Similarly

$$\|v\|_{D_s(R^N)}^2 \geq \mu_2^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Hence

$$A \geq \frac{s}{N} (\|u\|_{D_s(R^N)}^2 + \|v\|_{D_s(R^N)}^2) \geq (\mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}}) S_s^{\frac{N}{2s}}.$$

By lemma 2.12 in [22], we have

$$w_{\mu_i} = \left( \frac{S_s}{\mu_i} \right)^{\frac{1}{2^*-2}} \bar{u},$$

is the solution of the equation

$$(-\Delta)^s u = \mu_i |u|^{2^*-2} u \quad \text{in } R^N$$

and

$$w_{\mu_i} = (\mu_i)^{\frac{2s-N}{4s}},$$

satisfies

$$(-\Delta)^s u = \mu_i |u|^{2^*-2} u \quad \text{in } R^N \quad i = 1, 2.$$

Let  $e_1 = (1, 0, \dots, 0) \in R^N$ ,  $(U_R(X), V_R(X)) = (w_{\mu_1}(x), w_{\mu_2}(x + Re_1))$ , where  $R$  is a positive constant. Since  $V_R(x) \in L^{2^*}(R^N)$  is a solution of

$$(-\Delta)^s u = \mu_i |u|^{2^*-2} u \quad \text{in } R^N, \quad i = 1, 2,$$

we have,  $V_R(X) \rightharpoonup 0$  in  $L^{2^*}(R^N)$  as  $R \rightarrow +\infty$ , hence

$$\begin{aligned} \lim_{R \rightarrow +\infty} \int_{R^N} U_R^\alpha V_R^\beta dx &= \lim_{R \rightarrow +\infty} \int_{R^N} U_R^\alpha V_R^{\frac{\alpha}{2^*-1}} V_R^{\frac{2^*(\beta-1)}{2^*-1}} dx, \\ &\leq \lim_{R \rightarrow +\infty} \left( \int_{R^N} U_R^{2^*-1} V_R dx \right)^{\frac{\alpha}{2^*-1}} \left( \int_{R^N} V_R^{2^*} dx \right)^{\frac{\beta-1}{2^*-1}} \rightarrow 0. \end{aligned}$$

Therefore, for  $R > 0$  sufficiently large, the system



$$(4.1) \quad \begin{cases} \|U_R\|_{D_s(R^N)}^2 = \int_{R^N} \mu_1 |U_R|^{2^*} dx \\ \quad = t_R^{\frac{2^*-2}{2}} \int_{R^N} \mu_1 |U_R|^{2^*} dx + t_R^{\frac{\alpha-2}{2}} s_R^{\frac{\beta}{2}} \int_{R^N} \frac{\alpha\gamma}{2^*} U_R^\alpha V_R^\beta dx, \\ \|V_R\|_{D_s(R^N)}^2 = \int_{R^N} \mu_2 |V_R|^{2^*} dx \\ \quad = s_R^{\frac{2^*-2}{2}} \int_{R^N} \mu_2 |V_R|^{2^*} dx + t_R^{\frac{\alpha}{2}} s_R^{\frac{\beta-2}{2}} \int_{R^N} \frac{\alpha\gamma}{2^*} U_R^\alpha V_R^\beta dx, \end{cases}$$

has a solution  $(t_R, s_R)$  with

$$\lim_{R \rightarrow +\infty} (|t_R - 1| + s_R - 1) = 0 \quad (\star)$$

and

$$(\sqrt{t_R} U_R, \sqrt{s_R} V_R) \in \mathbb{N}.$$

Hence

$$\begin{aligned} A &= \inf_{(u,v) \in \mathbb{N}} E(u, v) \leq E(\sqrt{t_R} U_R, \sqrt{s_R} V_R), \\ &= \frac{s}{N} (t_R \|U_R\|_{D_s(R^N)}^2 + s_R \|V_R\|_{D_s(R^N)}^2), \\ &\leq \frac{s}{N} (t_R \mu_1^{-\frac{N-2s}{2s}} \|U_{\varepsilon,y}\|_{D_s(R^N)}^2 + s_R \mu_2^{-\frac{N-2s}{2s}} \|U_{\varepsilon,y}\|_{D_s(R^N)}^2). \end{aligned}$$

Let  $R \rightarrow +\infty$ , we get

$$A \leq \frac{s}{N} (\mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}}) S_s^{\frac{N}{2s}}.$$

Therefore

$$(4.2) \quad A = \frac{s}{N} (\mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}}) S_s^{\frac{N}{2s}}.$$

Suppose that A is attained by some  $(u, v) \in N$ , then  $E(u, v) = A$ . By Lemma 2.2, we know  $(u, v)$  is a nontrivial solution of (1.1). By Strong maximum principle for fractional laplacian (Proposition 2.17 in [9], Lemma 6 in [15]) and comparison principle in [13], we may assume that  $u > 0, v > 0$  and

$$\int_{R^N} |u|^\alpha |v|^\beta dx > 0,$$

then

$$\|u\|_{D_s(R^N)}^2 \leq \int_{R^N} \mu_1 |u|^{2^*} dx \leq \mu_1 S_s^{-\frac{2^*}{2}} (\|u\|_{D_s(R^N)}^2)^{\frac{2^*}{2}}.$$

Hence

$$\|u\|_{D_s(R^N)}^2 \geq \mu_1^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Similarly

$$\|v\|_{D_s(R^N)}^2 \geq \mu_2^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}.$$

Then

$$A = E(u, v) = \frac{S}{N}(\|u\|_{D_s(R^N)}^2 + \|v\|_{D_s(R^N)}^2) \geq \frac{S}{N}(\mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}})S_s^{\frac{N}{2s}},$$

which contradicts to (4.2).  $\square$

Finally we prove  $(\star)$

*Proof.* Let

$$\frac{\int_{R^N} U_R^\alpha V_R^\beta dx}{\int_{R^N} \mu_1 |U_R|^{2^*} dx} = a,$$

we have  $a \rightarrow 0$  as  $R$  sufficiently large. Then (4.1) has a solution is equivalent to that the following system has a solution at the neighbourhood of point  $(1, 1)$ .

$$\begin{cases} k^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} k^{\frac{\alpha-2}{2}} l^{\frac{\beta}{2}} a = 1, \\ l^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} a = 1, \end{cases}$$

then using Taylor expansion at  $(1, 1)$ , we have

$$\begin{cases} (1 + \frac{2^*-2}{2}(k-1) + O((k-1)^2)) \\ + \frac{\alpha\gamma}{2^*}(1 + \frac{\alpha-2}{2}(k-1) + O((k-1)^2))(1 + \frac{\beta}{2}(l-1) + O((l-1)^2))a = 1, \\ (1 + \frac{2^*-2}{2}(l-1) + O((l-1)^2)) \\ + \frac{\beta\gamma}{2^*}(1 + \frac{\alpha}{2}(k-1) + O((k-1)^2))(1 + \frac{\beta-2}{2}(l-1) + O((l-1)^2))a = 1. \end{cases}$$

Therefore

$$\begin{cases} k-1 = -\frac{\alpha\gamma}{2^*}(\frac{\alpha-2}{2^*-2})(k-1)a - \frac{\alpha\gamma}{2^*}(\frac{\beta}{2^*-2})(l-1)a - \frac{\alpha\gamma}{2^*}(\frac{2}{2^*-2})a \\ + O((k-1)^2 + (l-1)^2), \\ l-1 = -\frac{\beta\gamma}{2^*}(\frac{\alpha}{2^*-2})(k-1)a - \frac{\beta\gamma}{2^*}(\frac{\beta-2}{2^*-2})(l-1)a - \frac{\beta\gamma}{2^*}(\frac{2}{2^*-2})a \\ + O((k-1)^2 + (l-1)^2). \end{cases}$$

Then let

$$\begin{aligned} a_1 &= -\frac{\alpha\gamma}{2^*}(\frac{\alpha-2}{2^*-2}), b_1 = -\frac{\alpha\gamma}{2^*}(\frac{\beta}{2^*-2}), c_1 = -\frac{\alpha\gamma}{2^*}(\frac{2}{2^*-2})a, \\ a_2 &= -\frac{\beta\gamma}{2^*}(\frac{\alpha}{2^*-2}), b_2 = -\frac{\beta\gamma}{2^*}(\frac{\beta-2}{2^*-2}), c_2 = -\frac{\beta\gamma}{2^*}(\frac{2}{2^*-2})a, \end{aligned}$$

we have that

$$(4.3) \quad \begin{cases} k-1 = a_1 a(k-1) + b_1 a(l-1) + c_1 + O((k-1)^2 + (l-1)^2), \\ l-1 = a_2 a(k-1) + b_2 a(l-1) + c_2 + O((k-1)^2 + (l-1)^2). \end{cases}$$

Let

$$x = \begin{pmatrix} k-1 \\ l-1 \end{pmatrix},$$

$$B = \begin{pmatrix} a_1 a & b_1 a \\ a_2 a & b_2 a \end{pmatrix},$$

$$c = \begin{pmatrix} c_1 + O((k-1)^2 + (l-1)^2) \\ c_2 + O((k-1)^2 + (l-1)^2) \end{pmatrix},$$

then

$$x = Bx + c \triangleq T(x).$$

We claim that,  $T$  is a contraction mapping. In fact

$$\|T(x_1) - T(x_2)\| \leq \|B\|_2 \|x_1 - x_2\|,$$

where  $\|B\|_2 = \sqrt{\lambda}$ ,  $\lambda$  is maximum eigenvalue of  $B^T B$ , since  $a \rightarrow 0$ , there exists  $0 < \alpha < 1$ , s.t.,  $\|B\|_2 \leq \alpha$ . By mapping contraction theorem, we get that the systems (2.1) has a solution  $(t_R, s_R)$  with

$$\lim_{R \rightarrow +\infty} (|t_R - 1| + s_R - 1) = 0.$$

□

## 5. PROOF OF THEOREM 1.2

*Proof.* By Remark 2.2, we have

$$(5.1) \quad A \leq E(\sqrt{k_0}U_{\varepsilon,y}, \sqrt{l_0}U_{\varepsilon,y}) = \frac{s}{N}(k_0 + l_0)S_s^{\frac{N}{2s}}.$$

Let  $(u_n, v_n) \in \mathbb{N}$  be a minimizing sequence for  $A$ , that is  $E(u_n, v_n) \rightarrow A$  as  $n \rightarrow \infty$ .

Define

$$c_n = \left( \int_{R^N} |u_n|^{2^*} dx \right)^{\frac{2}{2^*}}, \quad d_n = \left( \int_{R^N} |v_n|^{2^*} dx \right)^{\frac{2}{2^*}},$$

then

$$\begin{aligned} S_s c_n &\leq \|u_n\|_{D_s(R^N)}^2 = \int_{R^N} \mu_1 |u_n|^{2^*} + \frac{\alpha\gamma}{2^*} |u_n|^\alpha |v_n|^\beta dx, \\ &\leq \mu_1 (c_n)^{\frac{2^*}{2}} + \frac{\alpha\gamma}{2^*} \left( \int_{R^N} |u_n|^{2^*} \right)^{\frac{\alpha}{2^*}} \left( \int_{R^N} |v_n|^{2^*} \right)^{\frac{\beta}{2^*}}, \\ &\leq \mu_1 (c_n)^{\frac{2^*}{2}} \frac{\alpha\gamma}{2^*} (c_n)^{\frac{\alpha}{2}} (d_n)^{\frac{\beta}{2}}, \end{aligned}$$

$$\begin{aligned} S_s d_n &\leq \|v_n\|_{D_s(R^N)}^2 = \int_{R^N} \mu_1 |v_n|^{2^*} + \frac{\alpha\gamma}{2^*} |u_n|^\alpha |v_n|^\beta dx, \\ &\leq \mu_2 (d_n)^{\frac{2^*}{2}} \frac{\beta\gamma}{2^*} (c_n)^{\frac{\alpha}{2}} (d_n)^{\frac{\beta}{2}}. \end{aligned}$$

Dividing both side of inequality by  $S_s c_n$  and  $S_s d_n$ .

Let

$$\tilde{c}_n = \frac{c_n}{S_s^{\frac{2^*-2}{2}}}, \quad \tilde{d}_n = \frac{d_n}{S_s^{\frac{2^*-2}{2}}},$$

we get

$$\begin{cases} \mu_1 \tilde{c}_n^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} \tilde{c}_n^{\frac{\alpha-2}{2}} \tilde{d}_n^{\frac{\beta}{2}} \geq 1, \\ \mu_2 \tilde{c}_n^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} \tilde{c}_n^{\frac{\alpha}{2}} \tilde{d}_n^{\frac{\beta-2}{2}} \geq 1, \end{cases}$$

that is

$$F_1(\tilde{c}_n, \tilde{d}_n) \geq 0, \quad F_2(\tilde{c}_n, \tilde{d}_n) \geq 0.$$

Consquently, when  $2s < N < 4s$ ,  $\alpha, \beta > 2$ , Proposition 4.1 ensures that

$$\tilde{c}_n + \tilde{d}_n \geq k + l = k_0 + l_0.$$

Therefore

$$c_n + d_n \geq (k_0 + l_0) S_s^{\frac{2}{2^*-2}} = (k_0 + l_0) S_s^{\frac{N-2s}{2s}}$$

Since  $(u_n, v_n) \in \mathbb{N}$  is a minimizing sequence for A, we have

$$(5.2) \quad E(u_n, v_n) = \frac{s}{N} (||u_n||_{D_s(R^N)}^2 + ||v_n||_{D_s(R^N)}^2).$$

By the definition of  $S_s$  (2.1) and (5.2) we have,

$$S_s(c_n + d_n) \leq (||u_n||_{D_s(R^N)}^2 + ||v_n||_{D_s(R^N)}^2) = \frac{N}{s} E(u_n, v_n) = \frac{N}{s} A + O(1).$$

Since

$$S_s(c_n + d_n) \geq (k_0 + l_0) S_s^{\frac{N}{2s}}, \quad A \leq \frac{s}{N} (k_0 + l_0) S_s^{\frac{N}{2s}},$$

we have

$$(k_0 + l_0) S_s^{\frac{N}{2s}} \leq S_s(c_n + d_n) \leq (k_0 + l_0) S_s^{\frac{N}{2s}},$$

that is

$$c_n + d_n \rightarrow (k_0 + l_0) S_s^{\frac{2}{2^*-2}} \text{ as } n \rightarrow +\infty,$$

thus

$$A = \lim_{n \rightarrow +\infty} E(u_n, v_n) \geq \lim_{n \rightarrow +\infty} \frac{s}{N} S_s(c_n + d_n) \geq \frac{s}{N} (k_0 + l_0) S_s^{\frac{N}{2s}}.$$

Hence

$$A = \frac{s}{N} (k_0 + l_0) S_s^{\frac{N}{2s}} = E(\sqrt{k_0} U_{\varepsilon, y}, \sqrt{l_0} U_{\varepsilon, y}).$$

□

## 6. PROOF OF THEOREM 1.3

*Proof.* To obtain the existence of  $(k(\gamma), l(\gamma))$  for  $\gamma > 0$ , we define functions

$$F_1(k, l, \gamma) = \mu_1 k^{\frac{2^*-2}{2}} + \frac{\alpha\gamma}{2^*} k^{\frac{\alpha-2}{2}} l^{\frac{\beta}{2}} - 1 \quad k, l > 0,$$

$$F_2(k, l, \gamma) = \mu_2 l^{\frac{2^*-2}{2}} + \frac{\beta\gamma}{2^*} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} - 1 \quad k, l > 0,$$

let

$$k(0) = \mu_1^{-\frac{2}{2^*-2}}, \quad l(0) = \mu_2^{-\frac{2}{2^*-2}},$$

then

$$\begin{aligned} F_1(k(0), l(0), 0) &= F_2(k(0), l(0), 0) = 0, \\ \partial_k F_1(k(0), l(0), 0) &= \frac{2^*-2}{2} \mu_1 k^{\frac{2^*-4}{2}} > 0, \\ \partial_l F_1(k(0), l(0), 0) &= \partial_k F_2(k(0), l(0), 0) = 0, \\ \partial_l F_2(k(0), l(0), 0) &= \frac{2^*-2}{2} \mu_2 l^{\frac{2^*-4}{2}} > 0. \end{aligned}$$

Denote

$$D = \begin{pmatrix} \partial_k F_1(k(0), l(0), 0) & \partial_l F_1(k(0), l(0), 0) \\ \partial_k F_2(k(0), l(0), 0) & \partial_l F_2(k(0), l(0), 0) \end{pmatrix},$$

since  $\det(D) > 0$ , by implicit function theorem, we see that  $k(\gamma), l(\gamma)$  are well defined and of class in  $C^1$  in  $(-\gamma_2, \gamma_2)$ , for some  $\gamma_2 > 0$  and

$$F_1(k(\gamma), l(\gamma), \gamma) = F_2(k(\gamma), l(\gamma), \gamma) = 0.$$

Thus  $(\sqrt{k_\gamma} U_{\varepsilon, y}, \sqrt{l_\gamma} U_{\varepsilon, y})$ , is a solution of (1.1).

Since

$$\lim_{\gamma \rightarrow 0} (k(\gamma) + l(\gamma)) = k(0) + l(0) = \mu_1^{-\frac{N-2s}{2s}} + \mu_2^{-\frac{N-2s}{2s}},$$

there exists  $\gamma_1 \in (0, \gamma_2)$  such that

$$k(\gamma) + l(\gamma) > \min\{\mu_1^{-\frac{N-2s}{2s}}, \mu_2^{-\frac{N-2s}{2s}}\} \quad \forall \gamma \in (0, \gamma_1).$$

By Lemma 6, we get

$$\begin{aligned} E((\sqrt{k_\gamma} U_{\varepsilon, y}, \sqrt{l_\gamma} U_{\varepsilon, y})) &= \frac{s}{N} (k(\gamma) + l(\gamma)) S_s^{\frac{N}{2s}}, \\ &> \min\left\{ \frac{s}{N} \mu_1^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}, \frac{s}{N} \mu_2^{-\frac{N-2s}{2s}} S_s^{\frac{N}{2s}} \right\} \\ &> A_\varepsilon = A = E(u, v). \end{aligned}$$

that is  $(\sqrt{k_\gamma} U_{\varepsilon, y}, \sqrt{l_\gamma} U_{\varepsilon, y})$  is another positive solution of (1.1).  $\square$

## REFERENCES

- [1] A.Capella, J.Pavila, L.Dupaigne and Y.Sire, Regularity of radial extremal solutions for some non-local semilinear equations, *Comm.Part Diff.Equal.*, (2011)1353-1384.
- [2] B.Barriours, E.Colorado, A.de Pablo and U.Sánchez, On some critical problems for the fractional Laplacian operator, *J.Differential Equations*, (2012)6133-6162
- [3] E.Colorado, A.de Pablo and U.Sánchez, On some critical problems for the fractional Laplacian operator, *J.Differential Equations*, (2012)6133-6162.
- [4] E.Colorado, A.de Pablo and U.Sánchez, Perturbation of a critical fractional equations, *Pacific J. Math.*, (2014)65-85.
- [5] F.Tao, X.Wu, Existence and multiplicity of positive solutions for fractional schodinger equations with critical growth, *Nonliner. Analysis*, (2017)158-174.
- [6] G.A.cotsiolis, N.K.Tavoularis, Best constant for sobolev inequality for higher order fractional derivatives, *J.Math.Anal.Appl.*, (2004) 225-236
- [7] J.Tan, The Brézis-Nirenberg type problem involving the square root of the Laplacian, *Calc. Var . Partical Differential Equations*, (2011) 21-41.
- [8] J.Marcos, D.Ferraz, Concentration-compactness principle for nonlocal scalar field equations with critical growth, *J.Math.Anal.Appl.*, (2017) 1189-1228.
- [9] L.Silvestre, Regularity of the obstacle problem for a fractional power of the Laplacian operator, *Comm.Pure Appl.Math.*, (2007)67-112
- [10] L.Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Comm.Partial Differential Equations*, (2007)1245-1260.
- [11] L.Caffarelli, J.Roquejoffre and Y.Sire, Variational problems for free boundaries for the fractional laplacian, *J.Eur.Math.Soc.*, (2010)1151-1179.
- [12] Q.Wang, Positive least energy solutions of fractional laplacian systems with critical exponent, *Electronic Journal of Differential Equations*, (2016)1-16.
- [13] Ros-Oton, Xavier, Nonlocal elliptic equations in bounded domains:a survey.*Publ.Mat.*, (2016)3-26.
- [14] R.Servadei, E.Valdinoci, The Brezis-Nirenberg result for the fractional laplacian, *Trans.Amer.Math.soc.* , (2015)67-102.
- [15] R.Servadei, E.Valdinoci, Weak and viscosity solution of the fractional Laplace equation, *Publ.Mat.*, (2014)133-154
- [16] X.Cabre, Y.Sure, Nonlinear equations for fractional laplacian,Regularity Maximum principle and Hamiltonian, *Ann.Inst.Henri poincare.Nonlincaire*, (2014)23-53.
- [17] X.He, M.Squassina and W.Zou, The Nehari manifold for fractional laplacian systems involving critical nonlinearities, *mathematical*, 2015
- [18] X.Zheng, J.Wang, Symmetry results for systems involving fractional Laplacian,*India.Journal of Pure and Applied Mathematics*, (2014)39-52
- [19] X.Shang, J.Zhang and Y.Yang, Positive solutions of nonhomogeneous fractional Laplacian problem with critical exponent,*Comm. Pure Appl.Anal.*, (2014) 567-584
- [20] Z.Chen, W.Zou, An optimal constant for the existence of least energy solution of a coupled Schrodinger system, *Cale.var Partial Differential Equation*, (2013)695-711.
- [21] Z.Chen, W.Zou, Positive least energy solutions and phase seperation for coupled Schrodinger equations with critical exponent, *Arch.Ration.Mech.Anal.*, (2012) 515-551.

- [22] Z.Guo, W.Zou, On critical systems involving fractional laplacian, *J.Math.Anal.Appl.*, (2017) 681-706.

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